The \textit{$p$-adic} integral geometry formula

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Joint work with Antonio Lerario
Part I: The real world
Integral geometry deals with averaging metric properties under the action of a Lie group.

Applications in: *representation theory, convex geometry, random algebraic geometry, etc.*
The metric
Let $S^n \to \mathbb{P}^n$ be the Hopf fibration. We define

$$d(x, y) = |\sin(\text{angle between the points})|$$

this descends to a metric on $\mathbb{P}^n$. The volume form on the sphere restricts to projective space.
The volume

If $Y$ is a codimension $m - k$ submanifold of a Riemannian manifold $M$ whose volume density is $\text{vol}$ and

$$U(Y, \epsilon) = \bigcup_{x \in Y} B(x, \epsilon)$$

denotes the $\epsilon$-neighborhood of $Y$ in $M$, then

$$\text{vol}_k(Y) := \lim_{\epsilon \to 0} \frac{\text{vol}(U(Y, \epsilon))}{\text{vol}(B_{\mathbb{R}^{m-k}}(0, \epsilon))}.$$
Theorem
Let $X, Y \subseteq \mathbb{P}^n(\mathbb{R})$ be real algebraic sets. Then

$$
\int_{SO_{n+1}(\mathbb{R})} \frac{\text{vol}_z(X \cap gY)}{\text{vol}_z\mathbb{P}^z} \, dg = \frac{\text{vol}_x X}{\text{vol}_x \mathbb{P}^x} \cdot \frac{\text{vol}_y Y}{\text{vol}_y \mathbb{P}^y}
$$

where $x = \dim X$, $y = \dim Y$, $z = \dim(X \cap gY)$.

Corollary
The expected number of zeros of a random real polynomial of degree $d$ is $\sqrt{d}$, with respect to a certain distribution.
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Act II: Leaving the real world.
Definition
The $p$-adic absolute value is defined on $\mathbb{Q}\backslash\{0\}$ by

$$|p^e a/b|_p := p^{-e} \quad e \in \mathbb{Z} \text{ and } \gcd(p, ab) = 1,$$

$$|0|_p := 0.$$

$$\mathbb{Q}_p = \left\{ \sum_{j \geq e} a_j p^j : e \in \mathbb{Z}, a_e \neq 0, a_j \in \{0, \ldots, p-1\} \right\}$$

with the ring structure given by “Laurent series arithmetic”.
|a + b|_p \leq \max\{|a|_p, |b|_p\}, \text{ with equality when this maximum is attained exactly once. (Ultrametric inequality)}.\\
|n|_p \leq 1 \text{ for all } n \in \mathbb{Z}.\\
|\cdot|_p : \mathbb{Q} \to \mathbb{R} \text{ has image } \{0\} \cup \{p^{-n} : n \in \mathbb{Z}\}.\\
(\mathbb{Q}_p, |\cdot|_p) \text{ is a metric space.}\\
For r \geq 0, define\\
\[ B(x; r) = \{ y \in \mathbb{Q}_p : |y - x|_p \leq r \}. \]
\(\mathbb{Q}_p\) is a locally compact topological space, and ring operations are continuous.

**Definition**
The *ring of p-adic integers* is defined as
\[ \mathbb{Z}_p := \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}. \]
The additive group \((\mathbb{Q}_p, +)\) is a locally compact topological group. Thus there is a Haar measure. We normalize \(\mu(\mathbb{Z}_p) = 1\).

**Lemma**

\(B(x; r)\) is closed and open in the metric topology. (When \(r > 0\))

**Lemma**

Let \(y \in B(x; r)\). Then \(B(x; r) = B(y; r)\).

**Corollary**

\(\mathbb{Z}_p\) is the disjoint union of the open and closed balls

\[
B(a; p^{-1}), \quad a \in \{0, 1, \ldots, p - 1\}.
\]

Therefore \(\mu(B(a; p^{-1})) = p^{-1}\), and \(\mu(\mathbb{Z}_p^\times) = 1 - p^{-1}\). (> 0 !!)
Critical detail: $\mathbb{Q}_p$ has a totally disconnected topology!!!
Define the \textit{padic unit sphere} by

\[ S^n_{\mathbb{Q}_p} := \{ x \in \mathbb{Q}_p^{n+1} : \|x\|_p = 1 \}. \]

\textbf{Remark}

The dimension of \( S^n \) is \( n + 1 \). Here is a picture:
\[ S^n_{\mathbb{Q}_p} := \{ x \in \mathbb{Q}_p^{n+1} : \| x \|_p = 1 \}. \]

We have the **Hopf fibration**

\[ \varphi: (a_0, \ldots, a_n) \mapsto (a_0 : \ldots : a_n) \]

The **spherical metric**

\[ d(x, y) := \| x \wedge y \|_p \]

gives \( \mathbb{P}^n(\mathbb{Q}_p) \) the structure of a metric space.

(\( \mathbb{R} \): the sine of the angle between \( x, y \in S^n_{\mathbb{R}} \) is \( \pm \| x \wedge y \|_\mathbb{R} \).)
**Definition**

\[ \mu(U) := \mu(\mathbb{Z}_p^\times)^{-1} \mu(\varphi^{-1}(U)). \]

**Proposition**

*A maximal compact subgroup of \( \text{GL}_{n+1}(\mathbb{Q}_p) \) is \( \text{GL}_{n+1}(\mathbb{Z}_p) \). It is unique up to conjugation.*

The metric on the unit sphere is \( \text{GL}_{n+1}(\mathbb{Z}_p) \)-invariant.
Definition

Let $X \subseteq \mathbb{P}^n$. For each $m$ define

$$N_m(X) := \frac{\# \{ x \pmod{p^m} : x \in \varphi^{-1}(X) \}}{p^m(1 - p^{-1})}.$$ 

The $d$-dimensional volume of $X \subseteq \mathbb{P}^n_{\mathbb{Q}_p}$ is

$$\text{vol}_d(X) := \lim_{m \to \infty} \frac{N_m(X)}{p^{md}}.$$ 

This definition of volume comes from the theory of zeta functions studied by Denef, Igusa, Oesterlé, Serre, etc.
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Proposition (Serre)

If $X$ is $d$-dimensional, then the limit in the definition exists and is finite.
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$$\text{vol}_d(X) := \lim_{m \to \infty} \frac{N_m(X)}{p^{md}}.$$ 

Lemma
This is also the $p$-adic tube volume. i.e,

$$\text{vol}_a(X) = \lim_{m \to \infty} p^{m(n-a)} \cdot \mu_n \left( \bigcup_{x \in X} B(x, p^{-m}) \right).$$
**Lemma**

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**Proof.**

Consider the continuous map

$$\pi_m: X \to \{ x \pmod{p^m} : x \in X \}.$$

Choosing an arbitrary set of lifts, we obtain a list of centers needed to cover $X$. 

\qed
Theorem
Let $\mathcal{X}$ be a subscheme of $\mathbb{P}^n$ which is smooth over $\text{Spec} \mathbb{Z}_p$. Then the $d$-dimensional volume is also the Weil canonical volume of $X = \mathcal{X}(\mathbb{Z}_p)$.

Proof.
By smoothness, the Jacobian is non-vanishing modulo $p$ at every point. We then use a quantitative inverse function theorem to count points modulo $p^m$.

$$\lim_{m \to \infty} \frac{N_m(X)}{p^{md}} = \frac{\mathcal{X}(\mathbb{F}_p)}{p^d}$$

This turns out to be equal to the Weil canonical volume. ∎
Examples in $\mathbb{P}^2(\mathbb{Q}_2)$:

$$\text{vol}_1 \mathcal{Z}(x - y + z) = \left(1 + \frac{1}{2}\right), \quad \text{vol}_1 \mathcal{Z}(x^2 + y^2 - z^2) = 1,$$

$$\text{vol}_1 \mathcal{Z}(2(y^2 - xz)) = \left(1 + \frac{1}{2}\right).$$
Act III: $p$-adic integral geometry
Theorem (K.-Lerario)

Let $X, Y \subseteq \mathbb{P}^n(\mathbb{Q}_p)$ be algebraic sets. Then

$$\int_{\text{GL}_{n+1}(\mathbb{Z}_p)} \frac{\text{vol}_z(X \cap gY)}{\text{vol}_z\mathbb{P}^z} \, dg = \frac{\text{vol}_x X}{\text{vol}_x \mathbb{P}^x} \cdot \frac{\text{vol}_y Y}{\text{vol}_y \mathbb{P}^y}$$

where $x = \dim X$, $y = \dim Y$, $z = \dim (X \cap gY)$. 
Proof sketch

Lemma (Hensel’s lemma)

Let \( f = (f_1, \ldots, f_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]^n \), let \( a \in \mathbb{Z}_p^n \), and let \( J_f(a) \) be the Jacobian matrix of \( f \) at \( a \). If

\[
\| f(a) \| < | J_f(a) |^2,
\]

then there is a unique \( \alpha \in \mathbb{Z}_p^n \) such that

\[ f(\alpha) = 0 \text{ and } \alpha \equiv a \pmod{p^m}, \]

where \( m := 1 - \log_p | J_f(a) |. \)
Lemma (Linear Approximation Lemma)

Let $X_1, \ldots, X_s \subseteq \mathbb{P}^n$ be algebraic sets such that
\[ \sum_{j=1}^s \operatorname{codim}(X_j) = n. \]
Let $x^{(j)} \in X_j$ be smooth points, and denote by $U_{x^{(j)}}$ balls of $\mathbb{P}^n$ of radius $p^{-m}$ centered at these points. Assume that in these balls we have local equations

\[ U_{x^{(j)}} \cap X_j = \{ f_{x^{(j)}} = 0 \}. \]

If $\left| J \left( f_1(x^{(1)}), \ldots, f_s(x^{(s)}) \right) \right|^2 > p^{-m}$, then

\[ \# \bigcap_{j=1}^s (X_j \cap U_{x^{(j)}}) = \# \bigcap_{j=1}^s (T_{x^{(j)}} X_j \cap U_{x^{(j)}}) = 0 \text{ or } 1. \]

Proof.

Apply Hensel’s lemma in each ball.
Proposition

For $A$ an open compact subset of an $a$-dimensional algebraic set in $\mathbb{P}_\mathbb{Q}_p^n$, we have

$$\int_{GL_{n+1}(\mathbb{Z}_p)} \frac{\text{vol}_k(A \cap gH)}{\text{vol}_k(\mathbb{P}_\mathbb{Q}_p^k)} \, dg = \frac{\text{vol}_a(A)}{\text{vol}_a(\mathbb{P}_\mathbb{Q}_p^a)}.$$
\[ \int_{GL_{n+1}(\mathbb{Z}_p)} \# (A \cap gL) \, dg \]

Convert a variety into a union of many tangent spaces (Linear approximation lemma)

\[ = \lim_{m \to \infty} \int_{GL_{n+1}(\mathbb{Z}_p)} \sum_{i=1}^{N_m(A)} \# (B(u_i, p^{-m}) \cap T_{u_i}A_i \cap gL) \]

Prove the result for pieces of linear varieties, then

\[ = \lim_{\ell \to \infty} N_m(A) \cdot \frac{\text{vol}_a(B(u, p^{-m}) \cap \mathbb{P}^a)}{\text{vol}_a(\mathbb{P}^a)} \]

Use the definition of volume

\[ = \frac{\text{vol}_a(A)}{\text{vol}_a(\mathbb{P}^a_{\mathbb{Q}_p})}. \]
Applications

Theorem (Oesterlé, weak form)

For an equidimensional variety $A \subseteq \mathbb{P}_{\mathbb{Q}_p}^n$ of dimension $d$, we have

$$N_m(A) \leq \deg(A) \cdot \text{vol}_d(\mathbb{P}_{\mathbb{Q}_p}^d) \cdot p^{md} + o(1).$$

Proof.

$$\int_{\text{GL}_{n+1}(\mathbb{Z}_p)} \frac{\text{vol}_k(A \cap gH)}{\text{vol}_k(\mathbb{P}_{\mathbb{Q}_p}^k)} \, dg = \frac{\text{vol}_a(A)}{\text{vol}_a(\mathbb{P}_{\mathbb{Q}_p}^a)}.$$

The integrand is at most the degree almost everywhere. \qed
Corollary

Let

\[ g(t) := \zeta_0 + \zeta_1 t + \ldots + \zeta_d t^d. \]

The expected number of zeros of \( g(t) \) in \( \mathbb{Q}_p \) is 1.

Proof.

Check that the standard Veronese is an isometry. Applying the Integral geometry formula gives the result. \( \square \)
Theorem (Evans, univariate)

Let

\[ g(t) := \zeta_0 + \zeta_1 \binom{t}{1} + \ldots + \zeta_d \binom{t}{d}, \]

where \( \{\zeta_k\}_{k=0}^d \) is a family of i.i.d. uniform variables in \( \mathbb{Z}_p \). Then the expected number of zeroes of \( g \) contained in \( \mathbb{Z}_p \) is

\[ p^{|\log_p d|} \left(1 + p^{-1}\right)^{-1}. \]

The expected number of zeros outside the unit disk is

\[ \frac{|d|}{p^{1} p^{-1}}. \]
Proof.
Let
\[ F: (t, 1) \rightarrow \left( 1, \binom{t}{1}, \ldots, \binom{t}{d} \right) \]
be the Mahler Veronese map. The Jacobian of \( F \) is
\[
\begin{bmatrix}
0, 1, \ldots, \frac{d}{dt} \binom{t}{d}
\end{bmatrix}.
\]
For \( t \in \mathbb{Z}_p \) the absolute value of the largest entry is exactly \( p^{\lfloor \log_p d \rfloor} \). The Hopf fibration restricted to the image of \( F \) is an isometry, so we can apply the integral geometry formula.
Thanks!

$p$-adic Integral Geometry, arXiv:1908.04775